Math 2471 – Spring 2010

Test 2 Review

For full credit: use calculus to solve problems, circle answers, and show all your work.

1) Find all critical numbers of:
   \[ f(x) = x^2(x + 5) \]
   \[ f'(x) = 3x^2 + 10x \]
   \[ 0 = 3x^2 + 10x \]
   \[ x = 0 \text{ or } x = -\frac{10}{3} \]

3) Find the value of \( c \) to match the mean value on \([0, \pi]\) of \( f(x) = \cos x \).
   \[ f'(c) = \frac{f(b) - f(a)}{b - a} \]
   \[ f'(c) = -\sin c \]
   \[ \cos c = \frac{-\sin c}{\pi - 0} \]
   \[ c = \frac{-\pi - 1}{2} \]

5) Below is a picture of a function. Sketch the graph of this function’s derivative on the same axes.

2) Find the value of \( c \) to match the mean value on \([-2, 2]\) of \( f(x) = x^3 \).
   Hint: mean value theorem!
   \[ f \text{ is continuous on } [-2, 2] \]
   \[ f(0) = 0 \]
   \[ f(2) = 8 \]
   \[ \frac{1}{2 - (-2)} \int_{-2}^{2} f(x) \, dx = 4 \]
   \[ \frac{1}{4} \cdot 2^3 = 8 \]

4) Find all maxima and minima on \( t < 3 \) of \( f(t) = t\sqrt{9 - t} \).
   \[ f(t) = t(9 - t) \]
   \[ f'(t) = 1 \]
   \[ f''(t) = -\frac{t}{\sqrt{9 - t}} \]
   \[ t \neq 9 \]
   \[ f'(t) = 0 \text{ at } t = 3 \]
   \[ f(3) = 3\sqrt{6} \]

6) Find all intervals on which the function \( f(x) = (x - 2)^2(x - 2) \) is increasing, decreasing, and all relative extrema.
   \[ f(x) = (x^2 - 4x + 4)(x - 2) \]
   \[ f'(x) = 3x^2 - 12x + 12 \]
   \[ f''(x) = 6x - 12 \]
   \[ f''(x) = 0 \text{ at } x = 2 \]
   Increasing: \( (-\infty, 2) \cup (2, \infty) \)
   Decreasing: \( (2, \infty) \)
   Extrema: None
   Max: \( \infty \)
   Min: \( -\infty \)
7) Find the points of inflection and discuss the concavity of \( f(x) = x^3 - 6x^2 + 12x \).

\[
\begin{align*}
\frac{d}{dx} f(x) &= 3x^2 - 12x + 12 \\
\frac{d^2}{dx^2} f(x) &= 6x - 12 \\
\end{align*}
\]

\( f''(1) = 0 \)

concave up

on \((2,\infty)\)

concave down

on \((0, 1)\)

on \((-\infty, 0)\)

\( f''(x) = 6x - 12 \\
0 = 6x - 12 \\
x = 2
\]

Inflection pt. @ \(x = 2\).  

8) Find the limit as \(x\) approaches infinity for \( f(x) = \frac{3x^2 - 8x}{x^2 + 2} \).

\[
\frac{d}{dx} f(x) = \frac{3x^2 - 8x}{x^2 + 2} \\
\frac{d^2}{dx^2} f(x) = \frac{3 - 8x^2}{(x^2 + 2)^2}
\]

\[
\lim_{x \to \infty} \frac{3 - 8x^2}{(1 + 2x^2)} = \frac{3 - 0}{1 + 0} = 3
\]

9) Find two positive numbers such that the second number is the reciprocal of the first and the sum is a minimum.

\[
\begin{align*}
x + \frac{1}{x} &= \text{Sum} \\
\frac{1}{x} &= \text{Reciprocal of } x \\
\frac{1}{x^2} &= \text{Minimum at } x = 1
\end{align*}
\]

\[
\text{Area of trapezoid} = \frac{100}{2} = 50 \\
\text{Height} = \frac{100}{w + 40} = \frac{100}{w} \\
200u^2 + 2w = \text{Minimum}
\]

10) Find the length and width of a rectangle with minimum perimeter and area of 100 sq. ft.

\[
\begin{align*}
\frac{d}{dx} f(x) &= x^3 + x + 1 \\
\frac{d^2}{dx^2} f(x) &= 3x^2 + 1
\end{align*}
\]

12) Use Newton's method to find the zero in \( f(x) = x^3 + x + 1 \). The formula Newton found was: \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \). Show each iteration.

13) Krusty the Clown is shot out of a circus cannon and his path can be modeled by the function \( f(t) = 5t^3 - 20t^2 + 20t \) where \( t \) is the time in seconds and \( f(t) \) is Krusty's height in meters for the first three seconds of his flight. Find Krusty's maximum and minimum heights.

\[
\begin{align*}
f'(t) &= 15t^2 - 40t + 20 \\
0 &= 15t^2 - 40t + 20 \\
0 &= 3t^2 - 8t + 4 \\
&= (3t - 2)(t - 2) \\
t &= \frac{2}{3}, 2
\end{align*}
\]

Max. 

Min. 

14) Write a story for the three seconds of Krusty's Krazy Kannon show. Include discussions of extrema, concavity, and inflection points.
15) Use the limit process to find the area under the curve of \( y = x^2 + 2 \) on \([2, 5]\).

\[
\lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left[ \left(2 + \frac{3i}{n}\right)^2 + 2 \right] = \lim_{n \to \infty} \frac{3}{n} \left[ \frac{18n}{n^2} + \frac{36}{n} + \frac{27n^2}{n^2} \right] \\
= \lim_{n \to \infty} \frac{3}{n} \left[ 18 + \frac{36}{n} + \frac{27n^2}{n^2} \right] = \frac{3}{6} \left[ 18 + 0 + 0 \right] = 9.
\]

16) I evaluated the integral \( \int_{2}^{5} (x^2) \, dx \) and found the result to be zero. I double checked my work and found no errors; however, I know there is some area between the graph and the x-axis. Please explain the result.

17) Evaluate the integral:

\[
\frac{1}{2} \int_{2}^{5} (-3y + 4) \, dy = \frac{-3}{2} \int_{2}^{5} y^2 + 4 \, dy + C \\
= \frac{-3}{2} \left[ \frac{y^3}{3} + 4y \right]_{2}^{5} = \frac{-3}{2} \left[ \frac{125}{3} - 8 \right] = \frac{-35}{2} + 20 = \frac{-35}{2} + \frac{40}{2} = \frac{5}{2}.
\]

18) Determine the area under the curve \( y = (3 - x) \sqrt{x} \) between \( x = 4 \) and \( x = 9 \).

\[
\int_{4}^{9} \sqrt{x} - \frac{3}{2} \sqrt{x} \, dx = \left[ \frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} \right]_{4}^{9} = \left[ \frac{27}{4} - \frac{32}{5} \right] = \left[ \frac{27}{4} - \frac{32}{5} \right] = \left[ \frac{135 - 128}{20} \right] = \frac{7}{20}.
\]

19) Evaluate the integral without using calculus or your calculator:

\[
\int_{0}^{4} 3x \, dx = \left[ \frac{3x^2}{2} \right]_{0}^{4} = \frac{3(16)}{2} - \frac{3(0)}{2} = 24.
\]

20) Find the indefinite integral and check the result by differentiation of \( \int (t^2 - \sin t) \, dt \).

\[
\int (t^2 - \sin t) \, dt = \frac{1}{3} t^3 + \cos t + C.
\]

Check:

\[
f(t) = \frac{1}{3} t^3 + \cos t + C \\
f'(t) = t^2 - \sin t.
\]
21) Use \( a(t) = -9.8 \text{ m/sec.}^2 \) due to gravity. Find an equation to represent the velocity and another equation to represent the height of an object using \( v_0 \) and \( h_0 \) for initial velocity and height.

Velocity equation:
\[
\frac{dv}{dt} = -9.8 \ x + c
\]
\[
v(t) = -9.8 \ x + v_0
\]

Height equation:
\[
f(x) = \frac{1}{2} \ a x^2 + v_0 \ x + h_0
\]

22) Evaluate the sum:
\[
\sum_{i=1}^{20} (i-1)^2
\]

23) Evaluate the integral:
\[
\int_0^1 (x - x^2) \, dx =
\]
\[
\left. \frac{1}{2} x^2 - \frac{1}{3} x^3 + C \right|_0^1
\]
\[
= \frac{1}{2} - \frac{1}{3} + C - (0 + 0 + C)
\]
\[
= \frac{1}{6}
\]

24) Evaluate the integral:
\[
\int_{-1}^1 (t^3 - 9t) \, dt =
\]
\[
\left. \frac{1}{4} t^4 - \frac{9}{2} t^2 + C \right|_{-1}^{1}
\]
\[
= \frac{1}{4} - \frac{9}{2} + C - \left( \frac{1}{4} (-1)^4 - \frac{9}{2} (-1)^2 + C \right)
\]
\[
= 0
\]

25) Find the indefinite integral:
\[
\int 5x^2 \sqrt{1 - x^2} \, dx =
\]
\[
-\frac{5}{2} \int -2x \sqrt{1 - x^2} \, dx
\]
\[
= -\frac{5}{2} \int u^{1/2} \, du
\]
\[
= -\frac{5}{2} \left( \frac{2}{3} u^{3/2} + C \right)
\]
\[
= -\frac{5}{6} \left( 1 - x^2 \right)^{3/2} + C
\]

26) Evaluate the integral:
\[
\int_0^\pi (1 + \sin x) \, dx =
\]
\[
= x + \cos x + C \bigg|_0^\pi
\]
\[
= \pi - \cos \pi + C - (0 + \cos 0 + C)
\]
\[
= \pi + 2
\]

27) Use Simpson's Rule to determine \( \int_0^a x^3 \, dx \) for \( n = 4 \).

\[
\int_0^a x^3 \, dx = \frac{a^4}{4} \left[ 0 + 4(\frac{a^3}{8}) + 2(1) + 4(\frac{a^2}{8}) + 8 \right]
\]
\[
= \frac{a^4}{12} \left[ 0 + 4 + 2 + \frac{a^2}{2} + 8 \right]
\]
\[
= \frac{a^4}{12} \left[ 10 + \frac{a^2}{2} \right]
\]
\[
= \frac{a^4}{2} \left( 5 + \frac{a^2}{4} \right)
\]
\[
= \frac{a^4}{2} \left( \frac{22}{4} \right)
\]
\[
= 4.25 \text{ sq. uni.}
\]